arXiv:hep-ph/9912485 v2 1 Mar 2001

Production of light pseudoscalars in external electromagnetic fields by the Schwinger mechanism

J. A. Grifols [a] , Eduard Massó [a] and Subhendra Mohanty [b]

[a] *Grup de Física Teòrica and IFAE, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain.*

[b] *Physical Research Laboratory, Navrangpura, Ahmedabad - 380 009, India.*

Abstract

We calculate the probability of the decay of external inhomogeneous electromagnetic fields to neutral pseudoscalar particles that have a coupling to two photons.

14.80.Mz, 11.30.Er,12.20.Ds,13.40.-f

Typeset using REVTEX

I. INTRODUCTION

The Schwinger mechanism is a non-perturbative process by which an infinite number of zero frequency photons can decay into electron-positron pairs [1]. In this paper we show that this mechanism can be generalized to study the production of other kinds of light particles from intense electromagnetic (EM) fields. The light particle that we consider is a pseudoscalar (PS) having a coupling to two photons.

In Section II we derive the formula for the decay of classical background fields into PS particles. This is achieved by integrating out the particle fields from the total Lagrangian to obtain the effective action of the classical background fields. The imaginary part of the effective Lagrangian is related to the probability of decay of classical background fields into particles. In Section III, we derive, from the usual coupling of the PS to two photons, the specific interaction Lagrangian that should be used in the general formalism of Section II in order to account for vacuum decay into PS. For static EM fields, we show that a necessary condition is that the fields are inhomogeneous. In Sections IV, V, and VI, we explicitly calculate the PS production in a variety of situations. Specifically we consider a cylindrical capacitor, a spherical capacitor, and a dipole magnetic field. A final section is devoted to the conclusions.

II. DECAY OF CLASSICAL BACKGROUND FIELDS INTO PARTICLES

We start with the action for the pseudoscalar ϕ (mass m) coupled to the background $\mathbf{E}$ and $\mathbf{B}$ fields of the general form

$$S[\phi, \mathbf{E}, \mathbf{B}] = \int d^4x \, \frac{1}{2}\phi(x) \left[-\partial^2 - m^2 + f(x)\right] \phi(x) \tag{1}$$

where $f(x)$ is some scalar function of $\mathbf{E}$ and $\mathbf{B}$ fields. From (1) we obtain the effective action for the background $\mathbf{E}$ and $\mathbf{B}$ fields formally as

$$e^{iS_{eff}[\mathbf{E},\mathbf{B}]} = \int \mathcal{D}\phi \; e^{iS[\phi,\mathbf{E},\mathbf{B}]} \tag{2}$$

The effective Lagrangian for the $\mathbf{E}$ and $\mathbf{B}$ fields can be related to the Green's function of ϕ in external $\mathbf{E}$ and $\mathbf{B}$ fields as follows. Differentiate (2) by m^2

$$\begin{aligned} i\frac{\partial S_{eff}[\mathbf{E},\mathbf{B}]}{\partial m^2} &= -\frac{\int \mathcal{D}\phi \; \phi^2 \; e^{iS[\phi,\mathbf{E},\mathbf{B}]}}{\int \mathcal{D}\phi \; e^{iS[\phi,\mathbf{E},\mathbf{B}]}} \\ &= -\frac{1}{2}\int d^4x \; G\left(x, x; \mathbf{E}, \mathbf{B}\right) \\ &= -\frac{1}{2}\int d^4x \; \int \frac{d^4p}{(2\pi)^4} \; G\left(p; \mathbf{E}, \mathbf{B}\right) \end{aligned} \tag{3}$$

The effective Lagrangian of the background fields is therefore formally given by the expression

$$\mathcal{L}_{eff}[\mathbf{E},\mathbf{B}] = \frac{i}{2}\int dm^2 \int \frac{d^4p}{(2\pi)^4} \; G\left(p; \mathbf{E}, \mathbf{B}\right) \tag{4}$$

The probability of external $\mathbf{E}$ and $\mathbf{B}$ fields to decay into quanta of ϕ is related to the imaginary part of $\mathcal{L}_{eff}$ as follows

$$\begin{aligned} P &= 1 - \left\langle 0 \left| \, e^{iS_{eff}[\mathbf{E},\mathbf{B}]} \, \right| 0 \right\rangle \\ &= 1 - \exp\left[-2 \, \mathrm{Im} \int d^3x \; dt \; \mathcal{L}_{eff}[\mathbf{E},\mathbf{B}] \right] \end{aligned} \tag{5}$$

In the case that this probability is small, we can write the probability density w (per unit volume and unit time) approximately as

$$w = 2 \; \mathrm{Im}\mathcal{L}_{eff}[\mathbf{E},\mathbf{B}] \tag{6}$$

We now give the general procedure for obtaining the effective action of the background fields by calculating the Green's function of ϕ in background $\mathbf{E}$ and $\mathbf{B}$ fields following the method of Duff and Brown [2].

The effective Lagrangian can be calculated by this method if the background fields contained in $f(x)$ in the interaction Lagrangian

$$\mathcal{L}_I(x) = \frac{1}{2} f(x)\phi^2(x) \tag{7}$$

can be expanded in a Taylor series near some reference point $\bar{x}$. Expanding $f(x)$ near $x = \bar{x}$,

$$f(x) = \alpha(\bar{x}) + \beta_\mu(\bar{x})(x-\bar{x})^\mu + \gamma^2_{\mu\nu}(\bar{x})(x-\bar{x})^\mu(x-\bar{x})^\nu + ... \tag{8}$$

$$\alpha(\bar{x}) = f(\bar{x}), \qquad \beta_\mu(\bar{x}) = \left(\frac{\partial f}{\partial x^\mu}\right)_{x=\bar{x}}, \qquad \gamma^2_{\mu\nu}(\bar{x}) = \frac{1}{2}\left(\frac{\partial}{\partial x^\mu}\frac{\partial f}{\partial x^\nu}\right)_{x=\bar{x}}$$

The equation for the Green's function for the ϕ field is given by

$$\left[\partial_x^2 + m^2 - \alpha - \beta_\mu(x-\bar{x})^\mu - \gamma^2_{\mu\nu}(x-\bar{x})^\mu(x-\bar{x})^\nu\right] G(x,\bar{x}) = \delta^4(x-\bar{x}) \tag{9}$$

In momentum space

$$(x-\bar{x})^\mu \to -i\frac{\partial}{\partial p_\mu} \tag{10}$$

and the equation for the Green's function in momentum space is

$$\left[-p^2 + m^2 - \alpha + i\beta_\mu\frac{\partial}{\partial p_\mu} + \gamma^2_{\mu\nu}\frac{\partial}{\partial p_\mu}\frac{\partial}{\partial p_\nu}\right] G(p) = 1 \tag{11}$$

We choose as an ansatz for the solution $G(p)$ the form

$$G(p) = i\int_0^\infty ds \; e^{-is(m^2-i\epsilon)} e^{ip_\mu A^{\mu\nu}p_\nu + B^\mu p_\mu + C} \tag{12}$$

where $A(s), B(s)$ and $C(s)$ are to be determined. They must satisfy the boundary condition in the case of vanishing external fields, i.e. when $\alpha, \beta, \gamma \to 0$

$$A^{\mu\nu} \to s\ g^{\mu\nu}, \qquad B^{\mu} \to 0, \qquad C \to 0 \tag{13}$$

and in this limit we should obtain

$$G(p) = i\int_0^\infty ds\ e^{-ism^2+isp^2} = \frac{1}{m^2-p^2} \tag{14}$$

i.e., the free particle Green's function.

To solve for A, B, and C we insert ansatz (12) in (11). We have

$$\begin{aligned} i\int_0^\infty ds\ \Big[-\mathrm{p}^2 + m^2 - \alpha + i\boldsymbol{\beta}\cdot(2i\mathrm{A}\cdot\mathrm{p}+\mathrm{B}) + (2i\mathrm{p}\cdot\mathrm{A}+\mathrm{B})\cdot\boldsymbol{\gamma}^2\cdot(2i\mathrm{A}\cdot\mathrm{p}+\mathrm{B}) \\ + 2i\ \mathrm{tr}(\boldsymbol{\gamma}^2\cdot\mathrm{A})\Big] \exp\Big\{-ism^2 + i\mathrm{p}\cdot\mathrm{A}\cdot\mathrm{p} + \mathrm{B}\cdot\mathrm{p} + \mathrm{C}\Big\} = 1 \end{aligned} \tag{15}$$

Equation (15) has the general form

$$\int_0^\infty ds\ g(s)\ e^{-h(s)} = 1 \tag{16}$$

whose solution is

$$g(s) = \frac{\partial h(s)}{\partial s} \tag{17}$$

with $h(0) = 0$ and $h(\infty) = \infty$. Using the form of the solution (17) for equation (15)

$$\begin{aligned} i\Big[-\mathrm{p}^2 + m^2 - \alpha + i\boldsymbol{\beta}\cdot(2i\mathrm{A}\cdot\mathrm{p}+\mathrm{B}) + (2i\mathrm{p}\cdot\mathrm{A}+\mathrm{B})\cdot\boldsymbol{\gamma}^2\cdot(2i\mathrm{A}\cdot\mathrm{p}+\mathrm{B}) \\ + 2i\,\mathrm{tr}(\boldsymbol{\gamma}^2\cdot\mathrm{A})\Big] = im^2 - i\mathrm{p}\cdot\frac{\partial\mathrm{A}}{\partial s}\cdot\mathrm{p} - \frac{\partial\mathrm{B}}{\partial s}\cdot\mathrm{p} - \frac{\partial\mathrm{C}}{\partial s} \end{aligned} \tag{18}$$

and comparing equal powers of p on both sides we get the following linear differential equations for A, B, and C,

$$\begin{aligned} \frac{\partial\mathrm{A}}{\partial s} &= 1 + 4\mathrm{A}\cdot\boldsymbol{\gamma}^2\cdot\mathrm{A} \\ \frac{\partial\mathrm{B}}{\partial s} &= 2i\mathrm{A}\cdot\boldsymbol{\beta} + 4\mathrm{A}\cdot\boldsymbol{\gamma}^2\cdot\mathrm{B} \\ \frac{\partial\mathrm{C}}{\partial s} &= i\alpha + \boldsymbol{\beta}\cdot\mathrm{B} - i\mathrm{B}\cdot\boldsymbol{\gamma}^2\cdot\mathrm{B} + 2\ \mathrm{tr}(\boldsymbol{\gamma}^2\cdot\mathrm{A}) \end{aligned} \tag{19}$$

The solutions of these equation which satisfy the boundary conditions (13) are given by

$$\mathrm{A} = \frac{1}{2}\boldsymbol{\gamma}^{-1}\cdot\tan(2\boldsymbol{\gamma}s) \tag{20}$$

$$\mathrm{B} = -\frac{i}{2}\boldsymbol{\gamma}^{-2}\cdot[1-\sec(2\boldsymbol{\gamma}s)]\cdot\boldsymbol{\beta} \tag{21}$$

$$\mathrm{C} = i\alpha s - \frac{1}{2}\mathrm{tr}\left[\ln\cos(2\boldsymbol{\gamma}s)\right] + \frac{i}{8}\boldsymbol{\beta}\cdot\boldsymbol{\gamma}^{-3}\cdot[\tan(2\boldsymbol{\gamma}s) - 2\boldsymbol{\gamma}s]\cdot\boldsymbol{\beta} \tag{22}$$

These A, B and C determine $G(p)$ when substituted in (12). The effective Lagrangian is obtained by substituting this $G(p)$ in (4) and carrying out the integration over m^2,

$$\mathcal{L}_{eff} = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\int \frac{d^4p}{(2\pi)^4}\ \exp\left\{-ism^2 + i\mathrm{p}\cdot \mathrm{A}\cdot \mathrm{p} + \mathrm{B}\cdot\mathrm{p} + \mathrm{C}\right\} \tag{23}$$

The Gaussian integral may be evaluated using

$$\int d^4p\ \exp\{i\mathrm{p}\cdot\mathrm{A}\cdot\mathrm{p} + \mathrm{B}\cdot\mathrm{p}\} = -i\pi^2\,(\det\mathrm{A})^{-\frac{1}{2}}\exp\left[\frac{i}{4}\ \mathrm{B}\cdot\mathrm{A}^{-1}\cdot\mathrm{B}\right] \tag{24}$$

where $\det\mathrm{A}$ is the determinant of the matrix A^μ_ν. Using now (20-22), we have

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\ e^{-is(m^2-\alpha)}\left[\det\left(\frac{2\boldsymbol{\gamma} s}{\sin 2\boldsymbol{\gamma} s}\right)\right]^{\frac{1}{2}} e^{il(s)} \tag{25}$$

where

$$l(s) = \frac{1}{4}\boldsymbol{\beta}\cdot\boldsymbol{\gamma}^{-3}\cdot[\tan(\boldsymbol{\gamma} s) - \boldsymbol{\gamma} s]\cdot\boldsymbol{\beta} \tag{26}$$

The coefficients of the Taylor expansion of the background fields (8) determine the effective action on integrating out the quantum field ϕ. In particular, an imaginary part of $\mathcal{L}_{eff}$ may be non-zero depending on the signs of the eigenvalues of the γ^2 matrix. When this occurrs, we have a non-zero probability (6) that the external EM fields decay in PS particles.

To the effective Lagrangian in (25) we should add subtractions to render it finite at $s=0$. When this is done, we have that in the limit $\beta\to 0$, $\gamma^2\to 0$, the effective Lagrangian $\mathcal{L}_{eff}\to 0$, as it should be. In the Appendix A we illustrate the method for the familiar case of production of charged scalar fields in a constant electric field.

The formulae (20-22,25) differ by some signs and factors of i from the solutions displayed in reference [3]. There, we presented the formulae for the case that $f(x)$ had only spatial variation and therefore β and γ^2 had only $i=1,2,3$ indices. In [3] we used the metric $(+,+,+)$ while in the present paper we consider spatial as well temporal variation and use the metric $(+,-,-,-)$. This introduces some changes in intermediate formulae but of course the final results we get in the present paper are identical with the final results we got in [3].

III. EFFECTIVE EM-FIELDS - PS PAIR INTERACTIONS

In this Section, we show that a coupling of a pseudoscalar to two photons induces an interaction $\mathcal{L}_I$ that may lead to PS production in a background of EM fields.

The generic pseudoscalar-two-photon interaction (see fig.1) can be written as

$$\mathcal{L}_{\phi\gamma\gamma} = \frac{1}{8}g\phi\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{27}$$

We need to evaluate the loop diagram of the type shown in fig.2 with infinite number of zero-frequency photon external legs. The imaginary part of this diagram gives the probability for the decay of the external electromagnetic field.

To calculate this diagram, we first evaluate the process $\phi A\to\phi A$, where A is an external photon. We use $i\mathcal{L}_{\phi\gamma\gamma}$ from (27) in momentum space,

$$\frac{1}{4}g\widetilde{\phi}\epsilon^{\mu\nu\rho\sigma}k_\mu\widetilde{A}_\nu\widetilde{F}_{\rho\sigma} \tag{28}$$

The two-photon two-PS interaction is then obtained contracting the internal photon legs,

$$4\left(\frac{1}{4}g\widetilde{\phi}\right)^2\epsilon^{\mu\nu\rho\sigma}k_\mu\widetilde{F}_{\rho\sigma}\frac{-ig_{\nu\nu'}}{k^2}\epsilon^{\mu'\nu'\rho'\sigma'}(-k_{\mu'})\widetilde{F}_{\rho'\sigma'} \tag{29}$$

The factor of 4 in equation (29) is for the four possible ways of joining the photon legs. Due to the presence of the k^2 term in the denominator, the effective coupling (29) is non-local. However, when we calculate the effective action for the external EM field the momentum k is integrated over. One can therefore make use of the identity

$$\int d^4k\ k_\mu k_{\mu'}\,g(k^2) = \int d^4k\ \frac{g_{\mu\mu'}k^2}{4}\,g(k^2) \tag{30}$$

to simplify (29). Thus, we can reduce the effective two PS-two photon interaction to a local interaction vertex. Back in configuration space, it is given by

$$\mathcal{L}_I = -\frac{1}{4}g^2\phi^2F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}g^2\phi^2(\mathbf{E}^2-\mathbf{B}^2) \tag{31}$$

(see fig.3).

With the interaction Lagrangian (31) we can go back to the formalism of Section II and calculate the probability density. We can readily identify $f(x)$ in (7),

$$f(x) = g^2(\mathbf{E}^2-\mathbf{B}^2) \tag{32}$$

In order to have a non trivial $\mathcal{L}_{eff}$, one needs non-zero second derivatives of the EM fields as they appear in expression (32). As we said in Section I, depending on the sign of the corresponding γ^2 matrix we may have PS production. We illustrate it in some simple physical situations in the following sections.

IV. PRODUCTION OF PSEUDOSCALARS IN A CYLINDRICAL CAPACITOR

The modulus of the electric field inside a cylindrical capacitor whose axis lies along the z-axis depends only on $\rho = (x^2+y^2)^{\frac{1}{2}}$,

$$E(\rho) = \frac{\lambda}{2\pi}\frac{1}{\rho} \tag{33}$$

with λ the linear electric charge density.

The bilinear interaction term (31) is

$$\begin{aligned}\mathcal{L}_I &= \frac{1}{2}g^2E^2(\rho)\phi^2(x)\\ &= \frac{1}{2}g_c^2\left(\frac{1}{\rho^2}\right)\phi^2(x)\end{aligned} \tag{34}$$

where $g_c \equiv \lambda g/2\pi$. The corresponding function $f(x)$ is

$$f(\rho) = g_c^2 \left(\frac{1}{\rho^2} \right) \tag{35}$$

Expanding the fields near some reference point (x_0, y_0, z_0) with $\rho_0 = (x_0^2 + y_0^2)^{\frac{1}{2}}$

$$\mathcal{L} = \frac{1}{2}\phi \left[-\partial^2 - m^2 + f(\rho_0) + \left.\frac{\partial f}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) + \frac{1}{2} \left.\frac{\partial^2 f}{\partial \rho^2}\right|_{\rho=\rho_0} (\rho - \rho_0)^2 \right] \phi + ... \tag{36}$$

It can be written as in (8) with

$$\alpha = f(\rho_0) \equiv \alpha_c \tag{37}$$

$$(\boldsymbol{\beta})_i = \frac{f'}{\rho_0}(x_0, y_0) \tag{38}$$

and

$$(\boldsymbol{\gamma}^{\mathbf{2}})_{ij} = \frac{f'}{2\rho_0^3} \begin{pmatrix} y_0^2 & -x_0 y_0 \\ -x_0 y_0 & x_0^2 \end{pmatrix} + \frac{f''}{2\rho_0^2} \begin{pmatrix} x_0^2 & x_0 y_0 \\ x_0 y_0 & y_0^2 \end{pmatrix} \tag{39}$$

where primes denote derivatives with respect to ρ taken at $\rho = \rho_0$. In the above formulae, the spatial indices run over $1, 2$.

Introducing the explicit form of f, we get

$$\alpha_c = \frac{g_c^2}{\rho_0^2} \tag{40}$$

$$(\boldsymbol{\beta})_i = -\frac{2g_c^2}{\rho_0^4}(x_0, y_0) \tag{41}$$

and the γ^2 matrix (39) reads

$$(\boldsymbol{\gamma}^{\mathbf{2}})_{ij} = \frac{g_c^2}{\rho_0^6} \begin{pmatrix} -y_0^2 + 3x_0^2 & 4x_0 y_0 \\ 4x_0 y_0 & -x_0^2 + 3y_0^2 \end{pmatrix} \tag{42}$$

Next, we diagonalise γ^2 by rotating the coordinates with an orthogonal matrix. For example, we can use

$$(\mathrm{O})_{ij} = \frac{1}{\rho_0} \begin{pmatrix} x_0 & y_0 \\ -y_0 & x_0 \end{pmatrix} \tag{43}$$

In diagonal form we have

$$(\boldsymbol{\gamma}_D^2)_{ij} = \frac{g_c^2}{\rho_0^4} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} a_c^2 & 0 \\ 0 & -b_c^2 \end{pmatrix} \tag{44}$$

We need $\vec{\beta}$ in the diagonal basis given by $\vec{\beta_D} = O \cdot \vec{\beta}$. We get

$$(\boldsymbol{\beta}_D)_i = \left(-2g_c^2\rho_0^{-3}, 0\right) \tag{45}$$

The expression for $\mathcal{L}_{eff}$ (25) finally reads

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\; e^{-is(m^2-\alpha_c)}\sqrt{\frac{2a_c s}{\sinh 2a_c s}}\;\sqrt{\frac{2b_c s}{\sin 2b_c s}}\; e^{il_c(s)} \tag{46}$$

where

$$l_c(s) = \lambda_c\,(a_c s - \tanh a_c s) \tag{47}$$

$$\lambda_c = g_c^4\rho_0^{-6}a_c^{-3} = \frac{g_c}{3\sqrt{3}} \tag{48}$$

One can perform the integration in (46) by extending s to the complex plane. The details of the integration are collected in Appendix B. We find that Im $\mathcal{L}_{eff}$ is given by the expression

$$\mathrm{Im}\,\mathcal{L}_{eff} = \frac{a_c^{\frac{3}{2}}b_c^{\frac{1}{2}}}{8\pi^2}\sum_{n=0}^{\infty}(-1)^n\,C_n^{(c)}\,e^{-\chi_c(2n+1)\pi} \tag{49}$$

$$C_n^{(c)} = \int_0^\pi du\;\frac{e^{-\chi_c u}e^{-\lambda_c\cot(u/2)}}{[u+(2n+1)\pi]^2\,[\sin u]^{\frac{1}{2}}\left[\sinh\left(\frac{b_c}{a_c}\,[u+(2n+1)\pi]\right)\right]^{\frac{1}{2}}} \tag{50}$$

where we can put $b_c/a_c = 1/\sqrt{3}$. We have defined

$$\chi_c = \frac{m^2-\alpha_c}{2a_c} + \frac{\lambda_c}{2} = \frac{1}{\sqrt{3}}\left(\frac{1}{2}\frac{m^2\rho_0^2}{g_c} - \frac{1}{3}\,g_c\right) \tag{51}$$

From (49) we can obtain the probability per unit volume and per unit time for PS production inside a cylindrical capacitor using (6). Keeping the leading $n=0$ term in (49) gives the probability density

$$w = \frac{3^{\frac{3}{4}}}{4\pi^2}\frac{g_c^2}{\rho_0^4}C_0^{(c)}\,e^{-\chi_c\pi} \tag{52}$$

V. PRODUCTION OF PSEUDOSCALARS IN A SPHERICAL CAPACITOR

The modulus of the electric field inside a spherical capacitor depends only on $r = |\vec{r}\,|$,

$$E(r) = \frac{Q}{4\pi}\frac{1}{r^2} \tag{53}$$

where Q is the electric charge.

The bilinear interaction term is then

$$\mathcal{L}_I = \frac{1}{2} g^2 E^2(r) \phi^2(x)$$
$$= \frac{1}{2} g_s^2 \left(\frac{1}{r^4} \right) \phi^2(x) \tag{54}$$

where $g_s = Qg/4\pi$.

The corresponding function $f(x)$ depends only on r,

$$f(r) = g_s^2 \left(\frac{1}{r^4} \right) \tag{55}$$

Expanding the fields near some reference point with (x_0, y_0, z_0) with modulus r_0

$$\mathcal{L} = \frac{1}{2} \phi \left[-\partial^2 - m^2 + f(r_0) + \left.\frac{\partial f}{\partial r}\right|_{r=r_0} (r - r_0) + \frac{1}{2} \left.\frac{\partial^2 f}{\partial r^2}\right|_{r=r_0} (r - r_0)^2 \right] \phi + ... \tag{56}$$

In Cartesian coordinates it can be written as in (8) with

$$\alpha = f(r_0) \equiv \alpha_s \tag{57}$$

$$(\boldsymbol{\beta})_i = \frac{f'}{r_0} (x_0, y_0, z_0) \tag{58}$$

and

$$(\boldsymbol{\gamma}^2)_{ij} = \frac{f'}{2r_0^3} \begin{pmatrix} y_0^2 + z_0^2 & -x_0 y_0 & -x_0 z_0 \\ -x_0 y_0 & x_0^2 + z_0^2 & -y_0 z_0 \\ -x_0 z_0 & -y_0 z_0 & x_0^2 + y_0^2 \end{pmatrix} + \frac{f''}{2r_0^2} \begin{pmatrix} x_0^2 & x_0 y_0 & x_0 z_0 \\ x_0 y_0 & y_0^2 & y_0 z_0 \\ x_0 z_0 & y_0 z_0 & z_0^2 \end{pmatrix} \tag{59}$$

where primes denote derivatives with respect to r taken at $r = r_0$. Introducing the form of $f(x)$, we get

$$\alpha_s = \left(\frac{g_s^2}{r_0^4} \right) \tag{60}$$

and

$$(\boldsymbol{\beta})_i = -\frac{4g_s^2}{r_0^6} (x_0, y_0, z_0) \tag{61}$$

and

$$(\boldsymbol{\gamma}^2)_{ij} = \frac{2g_s^2}{r_0^8} \begin{pmatrix} 5x_0^2 - y_0^2 - z_0^2 & 6x_0 y_0 & 6x_0 z_0 \\ 6x_0 y_0 & -x_0^2 + 5y_0^2 - z_0^2 & 6y_0 z_0 \\ 6x_0 z_0 & 6y_0 z_0 & -x_0^2 - y_0^2 + 5z_0^2 \end{pmatrix} \tag{62}$$

The γ^2 matrix (62) can be diagonalised by rotating the coordinates with an orthogonal matrix. For example, we can use

$$(\mathrm{O})_{ij} = \frac{1}{r_0 d_0} \begin{pmatrix} -z_0 r_0 & 0 & x_0 r_0 \\ -x_0 y_0 & d_0^2 & -y_0 z_0 \\ x_0 d_0 & y_0 d_0 & z_0 d_0 \end{pmatrix} \tag{63}$$

where $d_0 = \sqrt{x_0^2 + z_0^2}$.

In diagonal form we have

$$(\gamma_D^2)_{ij} = \frac{2g_s^2}{r_0^6}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix} \equiv \begin{pmatrix} -b_s^2 & 0 & 0 \\ 0 & -b_s^2 & 0 \\ 0 & 0 & a_s^2 \end{pmatrix} \tag{64}$$

One must also use $\vec{\beta}$ in the diagonal basis given by $\vec{\beta_D} = O \cdot \vec{\beta}$. We get

$$(\boldsymbol{\beta}_D)_i = \left(0, 0, -4g_c^2 r_0^{-5}\right) \tag{65}$$

The expression for $\mathcal{L}_{eff}$ (25) is given in this case by

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\, e^{-is(m^2-\alpha_s)}\sqrt{\frac{2a_s}{\sinh 2a_s s}\,\frac{2b_s s}{\sin 2b_s s}}e^{il_s(s)} \tag{66}$$

where

$$l_s(s) = \lambda_s\left(a_s s - \tanh a_s s\right) \tag{67}$$

$$\lambda_s = \frac{4g_s^4}{r_0^{10}}\frac{1}{a_s^3} = \frac{2}{5\sqrt{10}}\frac{g_s}{r_0} \tag{68}$$

One can perform the integration in (66) by extending s to the complex plane. Performing this integration (Appendix B), we find that Im $\mathcal{L}_{eff}$ is given by the expression

$$\mathrm{Im}\ \mathcal{L}_{eff} = \frac{b_s a_s}{8\pi^2}\sum_{n=0}^{\infty}(-1)^n C_n^{(s)} e^{-\chi_s(2n+1)\pi} \tag{69}$$

$$C_n^{(s)} = \int_0^\pi du\ \frac{e^{-\chi_s u}e^{-\lambda_s \cot(u/2)}}{[u+(2n+1)\pi]^{\frac{5}{2}}(\sin u)^{\frac{1}{2}}\sinh\left(\frac{b_s}{a_s}[u+(2n+1)\pi]\right)} \tag{70}$$

where we can put $b_s/a_s = 1/\sqrt{5}$. We have defined

$$\chi_s = \frac{m^2-\alpha_s}{2a_s} + \frac{\lambda_s}{2} = \frac{1}{2\sqrt{10}}\left(\frac{m^2 r_0^3}{g_s} - \frac{3}{5}\frac{g_s}{r_0}\right) \tag{71}$$

As before, we keep only the $n = 0$ term. The expression for the probability density is then

$$w = \frac{\sqrt{5}}{2\pi^2}\,\frac{g_s^2}{r_0^6}C_0^{(s)}e^{-\chi_s\pi} \tag{72}$$

VI. PRODUCTION OF PSEUDOSCALARS IN DIPOLE MAGNETIC FIELDS

In a static dipole magnetic field the PS-pair -EM interaction is given by

$$\begin{aligned}\mathcal{L}_I &= -\frac{1}{2}g^2 B^2(r)\phi^2 \\ &= -\frac{1}{2}g^2\left(B_0^2 z_0^6\,\frac{3z^2+r^2}{4r^8}\right)\phi^2\end{aligned} \tag{73}$$

where B_0 is the field strength at a point $\vec{r}_0=(0,0,z_0)$ on the z-axis. We have now

$$f(\vec{r}) = -g^2\left(B_0^2 z_0^6\,\frac{3z^2+r^2}{4r^8}\right) \tag{74}$$

Expanding $B^2(r)$ near the point $\vec{r}_0$

$$\mathcal{L} = \frac{1}{2}\phi\Big[-\partial^2 - m^2 + f(\vec{r}_0) + \left.\frac{\partial f}{\partial x_i}\right|_{\vec{r}=\vec{r}_0}(x_i - x_{i0}) \tag{75}$$

$$+\frac{1}{2}\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\vec{r}=\vec{r}_0}(x_i-x_{i0})(x_j-x_{j0})\Big]\phi + ... \tag{76}$$

we find that the coefficients of the Taylor expansion are given by

$$\alpha = f(\vec{r}_0) = -g^2 B_0^2 \equiv \alpha_m \tag{77}$$

$$(\boldsymbol{\beta})_i = \left.\frac{\partial f}{\partial x_i}\right|_{\vec{r}=\vec{r}_0} = (0,0,6g^2B_0^2 z_0^{-1}) \tag{78}$$

and

$$(\boldsymbol{\gamma^2})_{ij} = \frac{1}{2}\left.\frac{\partial^2 f}{\partial x_i\partial x_j}\right|_{\vec{r}=\vec{r}_0} = \frac{3g^2B_0^2}{4z_0^2}\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & -28\end{pmatrix} \equiv \begin{pmatrix}a_m^2 & 0 & 0\\ 0 & a_m^2 & 0\\ 0 & 0 & -b_m^2\end{pmatrix} \tag{79}$$

Therefore we find using the notation and formalism of Section II that the effective action on integrating out the PS field is given by

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\; e^{-is(m^2-\alpha_m)}\,\frac{2a_m s}{\sinh 2a_m s}\sqrt{\frac{2b_m s}{\sin 2b_m s}}e^{il_m(s)} \tag{80}$$

with

$$l_m(s) = \lambda_m(b_m s - \tan b_m s) \tag{81}$$

$$\lambda_m = \frac{9g^4B_0^4}{z_0^2}\frac{1}{b_m^3} = \frac{3}{7\sqrt{21}}gB_0z_0 \tag{82}$$

The imaginary part of the expression (80) can be performed by enclosing the simple poles at $s = -in\pi(2a_m)^{-1}$, $n = 0, 1, ...$, with a contour from below. We get

$$\mathrm{Im}\,\mathcal{L}_{eff} = \frac{1}{8\pi^{\frac{5}{2}}}\, a_m^{\frac{3}{2}}\, b_m^{\frac{1}{2}} \sum_{n=1}^{\infty}(-1)^{n+1} C_n^{(m)} e^{-\chi_m n\pi} \tag{83}$$

$$C_n^{(m)} = n^{-\frac{3}{2}} \left[\sinh n\frac{b_m}{a_m}\pi\right]^{-\frac{1}{2}} e^{\tilde{l}_m} \tag{84}$$

with

$$\chi_m = \frac{m^2 - \alpha_m}{2a_m} = \frac{1}{\sqrt{15}}\left(\frac{m^2 z_0}{gB_0} + z_0 g B_0\right) \tag{85}$$

and

$$\tilde{l}_m = \lambda_m \left[n\frac{b_m}{a_m}\frac{\pi}{2} - \tanh n\frac{b_m}{a_m}\frac{\pi}{2}\right] \tag{86}$$

In (84) and (86) we can put $b_m/a_m = \sqrt{28/5}$. The main contribution to the above integral comes from the $n = 1$ term and we find that w is given by the expression

$$w = \frac{3(5^{\frac{3}{4}})(28^{\frac{1}{4}})}{16}\frac{g^2 B_0^2}{z_0^2} C_1^{(m)} e^{-\chi_m \pi} \tag{87}$$

VII. CONCLUSIONS AND FINAL REMARKS

In the presence of strong external fields, the physical vacuum breaks down because particle-antiparticle pairs are being pumped out of it at the expense of field energy. The case of a strong uniform electric field spontaneously creating electron-positron pairs is the best known (QED) example for this phenomenon. Such process is of a non-perturbative nature and the QED case has been solved exactly by Schwinger and others [1]. Their solution however does not include the backreaction on the external field exerted by the presence of the produced e^+e^- pairs. Clearly, creation of pairs requires the supply of mass energy and kinetic energy which must be furnished by the external field. A balanced energy budget is therefore only possible through a corresponding reduction of the energy stored in the field. Because electrons and positrons carry charge they will fly to the external sources of the field and thus the field (and hence its energy) will diminish. So, unless from the outside the field is restored, the pair production process cannot be indefinitely sustained. If nothing is done from the outside a catastrophic breakdown of the initially strong (critical) electric field will inevitably follow.

In the present paper we dealt with pseudoscalar particles. Pseudoscalars are fundamental ingredients of many completions of Particle Physics models. Examples run from axions to superlight partners of gravitinos. In the previous sections we have derived the probability for pair production of PS in electric and magnetic fields. Contrary to the QED case mentioned above, constant fields do not cause the disruption of the vacuum. Field gradients are necessary for the phenomenon to occur. Hence, we studied PS pair production in inhomogeneous fields. We have calculated the probability in a general case and based our computation on an effective action formalism formulated by Brown and Duff. We then have applied the general formulae to a few specific cases: PS production between the plates of a charged capacitor (either cylindrical or spherical) and in a magnetic dipole field. Again, backreaction was ignored and therefore adequate boundary conditions were implicitly assumed that take into account the fact that pairwise creation of PS requires field energy to be depleted. Since our pseudoscalars are neutral, there is no possibility of charge flowing into the sources of external fields. Hence, what must be arranged for is that the geometry of the sources changes. For instance, the volume between plates of the capacitor should decrease. One may envisage, therefore, a quasi-static process where the capacitor plates adiabatically approach each other and where the Coulomb energy is partly stored in a mechanical device (an elastic spring, say) and the rest is lost in PS mass and PS kinetic energy. This process will end eventually at $t \to \infty$ when both plates meet and the charge on them is mutually neutralized. Our formula for the emission probability should be valid for such a setup (strictly adiabatic) for a finite period of time between $t = 0$ and $t = \infty$.

In the three cases studied, we found our probability to be of the generic form $w = ag^2\ exp(-bm^2/g - cg)$ in terms of the PS mass m and the PS-photon-photon coupling g (a, b, c are constants). At this point, we may mention that the special case where the PS is an axion introduces a simplification because m and g are related. Finally, we should point out that in a previous paper [3] we erroneously estimated axion emission in the Coulomb field of an atomic nucleus. This result is incorrect because we overlooked the question of appropriate boundary conditions that guarantee energy conservation and which are clearly not met in this microscopic system.

ACKNOWLEDGMENTS

We would like to thank G. Raffelt for pointing out a conceptual inconsistency in a previous version of this work and for the illuminating discussions that followed. We also thank K.V. Shajesh for his participation in the early stages of the paper. Two of us (J.A.G. and E.M.) have partial support from the CICYT Research Project AEN99-0766.

APPENDIX A: DECAY OF CONSTANT ELECTRIC FIELD INTO CHARGED SCALARS

We start with the equation for the Green's function $G(p)$

$$\begin{aligned}\left[-(p-eA)^2+m^2\right]G(p)&=1\\ \left[-p^2+m^2+e(A^\mu p_\mu+p_\mu A^\mu)-e^2A^2\right]G(p)&=1\end{aligned}\tag{A1}$$

We assume constant $\mathbf{E}$ and $\mathbf{B}$ fields. The vector potential can be choosen as

$$A_\mu=-\frac{1}{2}F_{\mu\nu}x^\nu\to\frac{i}{2}F_{\mu\nu}\frac{\partial}{\partial p_\nu}\tag{A2}$$

When inserted in (A1), one gets an equation of the form given in (11), except for a term

$$F^{\mu}{}_{\nu} p_{\mu} \frac{\partial}{\partial p_{\nu}} G(p) \tag{A3}$$

that leads to an expression containing

$$F^{\mu}{}_{\nu} p_{\mu} p_{\alpha} A^{\alpha\mu} \tag{A4}$$

The antisymmetry of F and the fact that F and A commute makes this term vanish. Then our equation is

$$\left[-p^2 + m^2 + \frac{e^2}{4} F_{\mu\nu} F^{\mu}_{\rho} \frac{\partial^2}{\partial p_{\nu} \partial p_{\rho}} \right] G(p) = 1 \tag{A5}$$

and with our definitions in (8), $\beta = 0$ and

$$\begin{aligned} \gamma^2_{\nu\rho} &= -\frac{e^2}{4} F_{\mu\nu} F^{\mu}_{\rho} \\ &\equiv -\frac{e^2}{4} F^2_{\nu\rho} \end{aligned} \tag{A6}$$

Let us work out the special case of a constant $\mathbf{E}$ field. We have

$$(\gamma^2)_{\nu\rho} = -\frac{e^2}{4} \begin{pmatrix} E^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -E^2 \end{pmatrix} \tag{A7}$$

(We have chosen the z-direction as the direction of $\mathbf{E}$). The eigenvalues of $\gamma^{2\,\nu}_{\ \ \rho}$ are negative so

$$\left[\det \left(\frac{2\boldsymbol{\gamma} s}{\sin 2\boldsymbol{\gamma} s} \right) \right]^{\frac{1}{2}} = \frac{eEs}{\sinh eEs} \tag{A8}$$

and

$$\mathcal{L}_{eff} = -\frac{1}{32\pi^2} \int_0^\infty \frac{ds}{s^3} \, e^{-ism^2} \frac{eEs}{\sinh eEs} \tag{A9}$$

To this expression for $\mathcal{L}_{eff}$ one should add a subtraction to make it finite at $s = 0$. When this is done, $\mathcal{L}_{eff} \to 0$ when $eE \to 0$.

The probability of scalar production can now be calculated using (5). The integral can be calculated by contour integration by closing the real axis with a contour on the negative imaginary plane. This contour encloses poles at $s = -in\pi/eE$ which contribute to the integral. The final result for the constant electric field decay probability density is

$$w = \frac{\alpha E^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp\left(-\frac{n\pi m^2}{eE} \right) \tag{A10}$$

which coincides with the well-known formula found in textbooks [4].

APPENDIX B: EVALUATION OF THE INTEGRAL

We are interested in evaluating the following integral

$$I = -\mathrm{Im}\ \frac{1}{32\pi^2}\int_0^{\infty}\frac{ds}{s^3}\sqrt{\frac{2as}{\sinh 2as}}\left(\frac{2bs}{\sin 2bs}\right)^{\frac{1}{k}} e^{-is(m^2-\alpha)}e^{il(s)} \tag{B1}$$

where

$$l(s) = \lambda\left(as - \tanh as\right), \qquad k = 1, 2. \tag{B2}$$

The above integral is an even function of s and thus can be written as

$$I = -\frac{1}{2}\mathrm{Im}\ \frac{1}{32\pi^2}\int_{-\infty}^{+\infty}\frac{ds}{s^3}\sqrt{\frac{2as}{\sinh 2as}}\left(\frac{2bs}{\sin 2bs}\right)^{\frac{1}{k}} e^{-is(m^2-\alpha)}e^{il(s)} \tag{B3}$$

Using the substitutions

$$2as = x, \qquad \gamma = \frac{b}{a}, \qquad \beta = \frac{m^2-\alpha}{2a} \tag{B4}$$

we have

$$I = -\frac{a^2}{16\pi^2}\mathrm{Im}\ \int_{-\infty}^{+\infty}\frac{dx}{x^3}\left(\frac{\gamma x}{\sin\gamma x}\right)^{\frac{1}{k}}\sqrt{\frac{x}{\sinh x}}e^{-i\beta x}e^{il(\frac{x}{2a})} \tag{B5}$$

Consider the integral

$$\int_C\frac{dz}{z^3}\left(\frac{\gamma z}{\sin\gamma z}\right)^{\frac{1}{k}}\sqrt{\frac{z}{\sinh z}}e^{-i\beta z}e^{il(\frac{z}{2a})} \tag{B6}$$

where C is the contour shown in the fig.4.

Clearly, what is needed is the integration around the branch cut along the negative imaginary axis. Furthermore, the branch cut segments from $-i2n\pi$ to $-i(2n+1)\pi$ contribute to the real part of the integral in $(B5)$ and the segments from $-i(2n+1)\pi$ to $-i2(n+1)\pi$ contribute to the imaginary part of the integral in $(B5)$. Thus, we have contributions of the branch cut from $-i\pi$ to $-i2\pi$, next we have from $-i3\pi$ to $-i4\pi$, and so on. The integral is evaluated as a summation over these contributions.

Let us evaluate the contribution of the first contour going around $-i\pi$ to $-i2\pi$,

$$\int_{-2i\pi+\epsilon}^{-i\pi+\epsilon} + \int_{-i\pi-\epsilon}^{-2i\pi-\epsilon}\frac{d(iy)}{(iy)^3}\left(\frac{(iy)}{\sin\gamma(iy)}\right)^{\frac{1}{k}}\sqrt{\frac{(iy)}{\sinh(iy)}}e^{-i\beta(iy)}e^{il(\frac{iy}{2a})} \tag{B7}$$

which under the substitution $u = y - \pi$ takes the form

$$2\int_0^{\pi} du\ \frac{e^{-\beta(u+\pi)}e^{-il(\frac{u+\pi}{2a})}}{(u+\pi)^{\left(\frac{5}{2}-\frac{1}{k}\right)}\left[\sinh\gamma(u+\pi)\right]^{\frac{1}{k}}\sqrt{\sin u}} \tag{B8}$$

which is equal to

$$2e^{-\chi\pi}\int_0^\pi du\ \frac{e^{-\chi u}e^{-\lambda\cot\frac{u}{2}}}{(u+\pi)^{\left(\frac{5}{2}-\frac{1}{k}\right)}(\sin u)^{\frac{1}{2}}\left[\sinh\gamma\,(u+\pi)\right]^{\frac{1}{k}}} \tag{B9}$$

with $\chi=\frac{m^2-\alpha}{2a}+\frac{\lambda}{2}$.

Evaluation of other contours can be done exactly as above. We have the final result to be

$$\mathrm{Im}\ \mathcal{L}_{eff}=\frac{a^{\left(2-\frac{1}{k}\right)}b^{\frac{1}{k}}}{8\pi^2}\sum_{n=0}^{\infty}(-1)^n C_n e^{-\chi(2n+1)\pi} \tag{B10}$$

$$C_n=\int_0^\pi du\ \frac{e^{-\chi u}e^{-\lambda\cot\frac{u}{2}}}{\left[u+(2n+1)\pi\right]^{\left(\frac{5}{2}-\frac{1}{k}\right)}(\sin u)^{\frac{1}{2}}\left[\sinh\gamma\,(u+(2n+1)\pi)\right]^{\frac{1}{k}}} \tag{B11}$$

REFERENCES

[1] J. Schwinger, *Phys. Rev.* **82** (1951) 664; for early work look in W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields*, Springer-Verlag (1985).

[2] M. R. Brown and M. J. Duff, *Phys. Rev.* **D11** (1975) 2124.

[3] J. A. Grifols, E. Massó, S. Mohanty and K.V. Shajesh, *Phys. Rev. D60* (1999) 097701.

[4] See e.g. C. Itzykson and J.B. Zuber, *Quantum Field Theory*, McGraw-Hill Co. (1980).

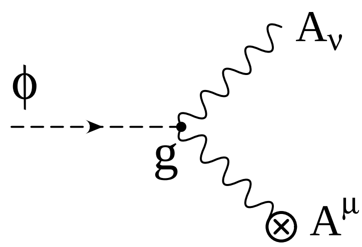

FIG. 1. PS-two-photon interaction.

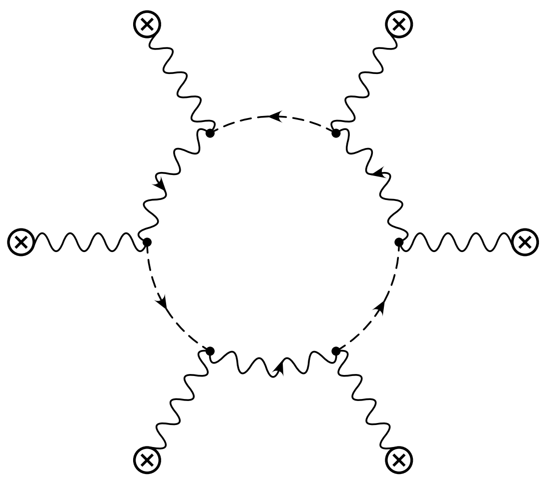

FIG. 2. Loop diagram showing infinite number of photon external legs.

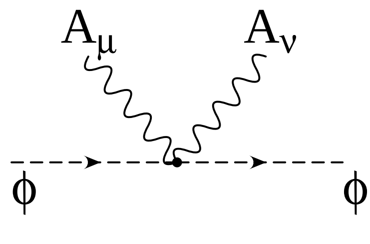

FIG. 3. Two PS-two photon interaction.

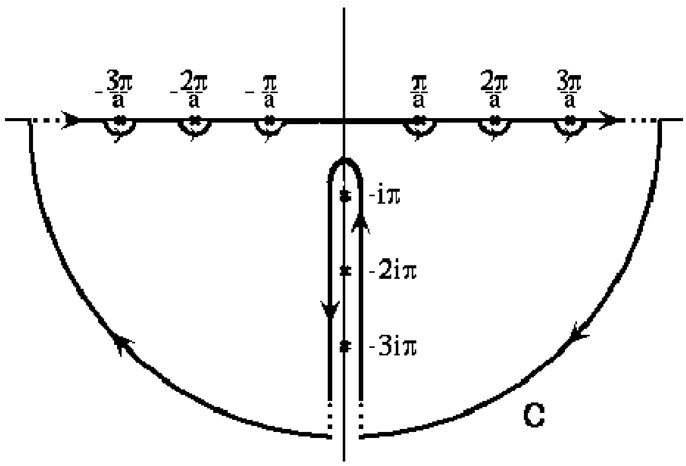

FIG. 4. Contour for evaluating the integral in (B6).